Khovanov-type homologies of null homologous links in $\mathbb{RP}^3$

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In S^3:

- $K \subseteq S^3 \backslash \{pt\} \cong \mathbb{R}^2 \times I$.
- link projection $L \subseteq \mathbb{R}^2$.
- Jones polynomial $V(L)$.
- Khovanov homology $Kh(L)$.
- Spectral sequence $Kh(m(L)) \Rightarrow \widehat{\mathrm{HF}}(\Sigma(S^3, K))$.

In $\mathbb{RP}^3$:

- $K \subset \mathbb{RP}^3 \backslash \{pt\} \cong \mathbb{RP}^2 \tilde{\times} I, [K] = 0 \in H_1(\mathbb{RP}^3, \mathbb{Z})$.
- link projection $L \subseteq \mathbb{RP}^2$.
- Kauffman bracket $\langle L \rangle$.
- Khovanov-type homology $\widetilde{\mathrm{Kh}}^{\alpha}(L)$ given by the E^2-page.
- Similar spectral sequence for null-homologous links in $\mathbb{RP}^3$.

Main results

Theorem (C.)

Let K be a null homologous link in $\mathbb{RP}^3$. There is a spectral sequence converging to to $\widehat{HF}(\Sigma_0(\mathbb{RP}^3, K))$, whose E^2 term consists of the Khovanov-type homology $\widetilde{Kh}^{\alpha_{HF}}(m(K))$.

Definition

A **dyad** is a tuple $\alpha = (V_0, V_1, f, g)$, $V_0 \underset{g}{\overset{f}{\rightleftarrows}} V_1$ such that

$f \circ g = 0, \ g \circ f = 0.$

Theorem (C.)

For each dyad $\alpha = (V_0, V_1, f, g)$, the homology $\widetilde{Kh}^{\alpha}(L)$ is an invariant of null homologous links in $\mathbb{RP}^3$.

Outline of the talk

- The spectral sequence relating $\widehat{HF}(\Sigma(S^3, L))$ and $\widetilde{Kh}(m(L))$ for links in S^3.
- Extend the spectral sequence for null homologous links in $\mathbb{RP}^3$.
- Combinatorial description of the E^2 pages, extending Khovanov homology to null-homologous links in $\mathbb{RP}^3$.

Link surgery spectral sequence of $\widehat{HF}$

For a knot K in a 3-manifold Y, a framing h of K is a choice of longitude in $\partial N(K)$. $Y_h(K)$ is the 3-manifold obtained by $Y \backslash N(K) \cup_f S^1 \times D^2$.

For a h-framed knot K in a 3-manifold Y, $(Y, Y_h(K), Y_{h+m}(K))$ forms a triad of 3-manifolds, and we have a long exact sequence.

$$... \to \widehat{HF}(Y) \to \widehat{HF}(Y_h(K)) \to \widehat{HF}(Y_{h+m}(K)) \to ...$$

In other words, $\widehat{CF}(Y)$ is quasi-isomorphic to the mapping cone $\widehat{f} : \widehat{CF}(Y_h(K)) \to \widehat{CF}(Y_{h+m}(K))$.

Link surgery spectral sequence of $\widehat{HF}$

For a h-framed link L of n components in a 3-manifold Y, we can apply a similar construction. For each $I \in \{0,1\}^n$, $Y(I)$ is the 3-manifold obtained from Y by applying h_j-framed surgery along L_j if $I_j = 0$, and $(h_j + m_j)$-framed surgery along L_j if $I_j = 1$.

Theorem ('05,Ozsváth, Szabó)

There is a spectral sequence whose E^1 term is

$$\bigoplus_{I \in \{0,1\}^n} \widehat{HF}(Y(I)),$$

which converges to $\widehat{HF}(Y)$.

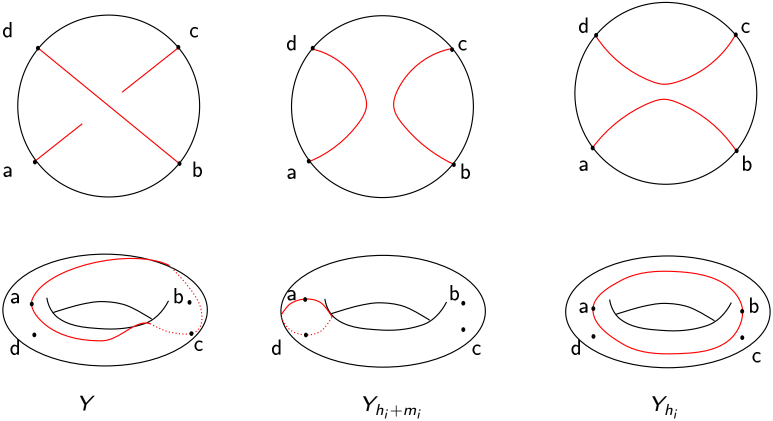
Branched double covers of S^3

Let K be a link in S^3. Denote the branched double cover of S^3 branching over K by $\Sigma(S^3, K)$.

We will choose a framed link L in $Y = \Sigma(S^3, K)$, such that $Y(I)$ corresponds to the branched double cover $\Sigma(S^3, K_I)$, where K_I is I-smoothing of K.

L is defined by the following. For the i-th crossings of K, the branched double cover of the vertical arc gives a component L_i of L.

The branched double cover of B^3 branching over 2 arcs is a solid torus. Different resolutions of $L \rightarrow$ Different ways to glue solid torus to get the branched double covers.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$Y_{h_i+m_i}$
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Relation with Khovanov homology

Each $Y(I)$ is a branched double cover of S^3 branching over an unlink K_I, which is $\#^{(k-1)}S^2 \times S^1$ if K_I has k-components.

$\widehat{HF}(\#^{(k-1)}S^2 \times S^1) = V^{\otimes(k-1)}$, where $V = \langle v_+, v_- \rangle$. This is exactly the vector space we associate to the unlink of k components in the reduced Khovanov homology. We can identify these two vector spaces with a canonical isomorphism, given by the basepoint on the link.

What's more, the d_1 map in the link spectral sequence of $\widehat{HF}$ corresponds to the differential map in the reduced Khovanov chain complex $\widetilde{CKh}$ under this canonical isomorphism.

To be more precise, it is the differential map in $\widetilde{CKh}(m(L))$, where $m(L)$ is the mirror of L. Hence, we obtain the following theorem:

Theorem ('05, Ozsváth, Szabó)

Let $K \subset S^3$ be a link. There is a spectral sequence whose E^2 terms consists of $\widetilde{CKh}(m(K))$, which converges to $\widehat{HF}(\Sigma(S^3, K), \mathbb{F}_2)$.

Branched double covers of 3−manifolds

For a link K in a 3-manifold M, branched double covers $\Sigma_h(M, K)$ are classified by the set of maps

$$\{h : H_1(M \backslash K, \mathbb{Z}) \longmapsto \mathbb{F}_2 \mid h([m_i]) = 1\},$$

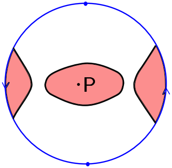
where m_i is the meridian of the i-th component of L.
Using this, we get the following results about $\Sigma_h(\mathbb{RP}^3, L)$:

- If K is nontrivial in $H_1(\mathbb{RP}^3, \mathbb{Z}_2)$, no branched double cover of $\mathbb{RP}^3$ branching over K.
- If K is null-homologous, then there are two branched double covers $\Sigma_h(\mathbb{RP}^3, K)$, determined by $h([r])$ for some $[r] \notin \langle [m_1], [m_2], ..., [m_n] \rangle$. Denote the one corresponding to $h([r]) = 0$ by $\Sigma_0(\mathbb{RP}^3, K, r)$.

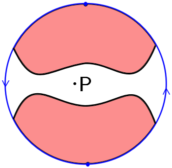
Link projections in $\mathbb{RP}^2$

Resolutions of null-homologous link projection L consists of null homologous circles in $\mathbb{RP}^2$, each dividing $\mathbb{RP}^2$ into a disk and a Möbius band.

For a point $P \in \mathbb{RP}^2 \backslash L$, we say P is encircled by a null homologous circle in $\mathbb{RP}^2$ if it lies in the disk bounded by the circle. Define $e_s(P)$ as the number of circles in L_s encircling P mod 2.



$e_s(P) = 1$



$e_s(P) = 0$

Branched double covers of $\mathbb{RP}^3$ over unlinks

Let C_P denote the circle in $\mathbb{RP}^3$, which is the union of the fiber of $\mathbb{RP}^3 \backslash \{*\} = \mathbb{RP}^2 \tilde{\times} I$ over P with $*$.

Lemma

Let L be a link projection in $\mathbb{RP}^2$, where each smoothing L_s consists of k_s unknots, then we have

$$\Sigma_0(\mathbb{RP}^3, L_s, C_P) = \begin{cases} (\mathbb{RP}^3 \# \mathbb{RP}^3) \# (S^1 \times S^2)^{\#(k_s - 1)} & \text{if } e_s(P) = 0, \\ (S^1 \times S^2)^{\# k_s} & \text{if } e_s(P) = 1. \end{cases}$$

The spectral sequence in $\mathbb{RP}^3$

For a link projection L in $\mathbb{RP}^2$, pick a point $P \in \mathbb{RP}^2 \backslash L$. We can use similar construction to form a link spectral sequence using the branched double covers $\Sigma(\mathbb{RP}^3, L_s, C_P)$.

The corresponding Heegaard Floer homology of these branched double covers are

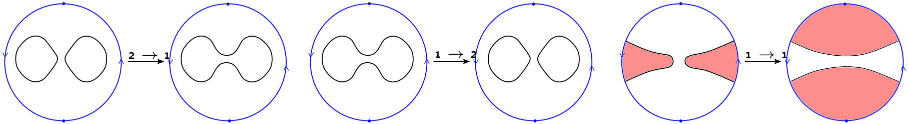
$$\widehat{HF}(\Sigma_0(\mathbb{RP}^3, L_s, C_P)) = \begin{cases} W \otimes V^{\otimes(k_s-1)} & \text{if } e_s(P) = 0, \\ \overline{V} \otimes V^{\otimes(k_s-1)} & \text{if } e_s(P) = 1, \end{cases}$$

where $W = \widehat{HF}(\mathbb{RP}^3 \# \mathbb{RP}^3) = \langle a, b, c, d \rangle$, $V = \widehat{HF}(S^1 \times S^2) = \langle v_+, v_- \rangle$ and $\overline{V} = \widehat{HF}(S^1 \times S^2) = \langle \overline{v}_+, \overline{v}_- \rangle$. These give the E^1 page of the spectral sequence.

The map d_1 in the spectral sequence

In the spectral sequence, the map d_1 corresponds to perform a knot surgery to the component corresponding to a crossing in the link projection. The effect on the 3-manifold is to change $\Sigma(\mathbb{RP}^3, L_s, C_P)$ to $\Sigma(\mathbb{RP}^3, L'_s, C_P)$, where $s' \in \{0, 1\}^n$ is differed from s in one slot, changing 1 to 0.

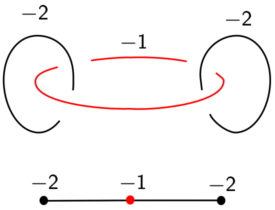
There are 3 such kinds of bifurcations for link projections in $\mathbb{RP}^2$.



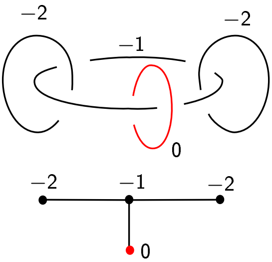
The maps d_1 associated to the $2 \to 1$ and $1 \to 2$ bifurcations are similar to those in the spectral sequence for links in S^3, which corresponds to the differential map in the reduced Khovanov chain complex.

The $1 \to 1$ bifurcation is special for links projections in $\mathbb{RP}^2$. They represents the surgery cobordisms $\mathbb{RP}^3 \# \mathbb{RP}^3 \to S^1 \times S^2$ or $S^1 \times S^2 \to \mathbb{RP}^3 \# \mathbb{RP}^3$, depending on the value $e_s(P)$.

The corresponding Kirby diagrams are as following.



(a)



(b)

Calculations in $\widehat{HF}$ gives the following:

Proposition

For the cobordism Z_a associated to (a), the induced map on $\widehat{HF}$ is

$$f = F_{Z_a} : \widehat{HF}(\mathbb{RP}^3 \# \mathbb{RP}^3) \longmapsto \widehat{HF}(S^1 \times S^2)$$
$$f(b) = f(c) = \overline{v}_-, \; f(a) = f(d) = 0.$$

For the cobordism Z_b associated to (b), the induced map on $\widehat{HF}$ is

$$g = F_{Z_b} : \widehat{HF}(S^1 \times S^2) \longmapsto \widehat{HF}(\mathbb{RP}^3 \times \mathbb{RP}^3)$$
$$g(\overline{v}_+) = b + c, \; g(\overline{v}_-) = 0.$$

In particular, $f \circ g = 0, g \circ f = 0$.

The main theorem

Theorem (C.)

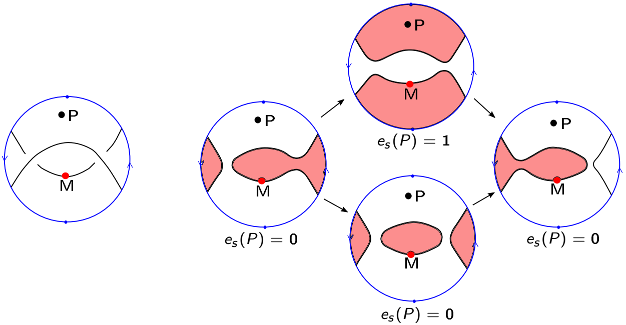
Let K be a null homologous link in $\mathbb{RP}^3$. There is a spectral sequence converging to to $\widehat{HF}(\Sigma_0(\mathbb{RP}^3, K))$, whose E^2 term consists of the Khovanov-type homology $\widetilde{Kh}^{\alpha_{HF}}(m(K))$.

Combinatorial description of E_2 page

Given a link projection $L \subseteq \mathbb{RP}^2$, pick a point M on the link projection L, and pick a point P in the complement $\mathbb{RP}^2 \backslash L$.

Take another input, a dyad $\alpha = (V_0, V_1, f, g)$, $V_0 \underset{g}{\overset{f}{\rightleftarrows}} V_1$ such that $f \circ g = 0$, $g \circ f = 0$. Then we can define the chain complex $(\widetilde{CKh}^{\alpha}(L), d)$ as usual (reduced) Khovanov chain complex.

For the E^2-page, the dyad is $\alpha_{HF} = (W, \bar{V}, f, g)$.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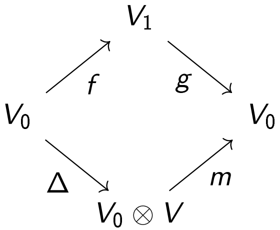
V_1
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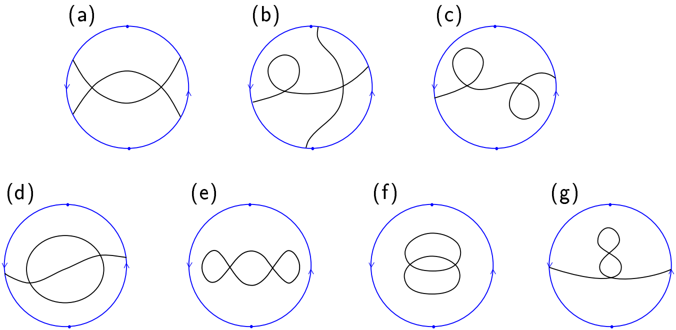
$d^2 = 0$

It is enough to check link projections with 2 crossings.



The algebraic relations we need are:

- V_0, V_1 are trivial V-bimodules;
- $f \circ g = 0, g \circ f = 0$.

Well-definedness of the homology

Proposition

The homology $\widetilde{Kh}^{\alpha}(L)$ is a link invariant for null homologous links in $\mathbb{RP}^3$.

- Choice of M: Different choices induce chain automorphism.
- Choice of P: Divide $\mathbb{RP}^2 \backslash L = R_0 \sqcup R_1$ according to linking number of C_p with L. Pick P in R_0.
- Different projections: Check invariance under Reidemeister moves in $\mathbb{RP}^2$.

Therefore, we get a link homology $\widetilde{Kh}^{\alpha}(K)$ for null homologous links in $\mathbb{RP}^3$.

The homology defined in [APS] corresponds to $\widehat{HF}^{\alpha}$ with $\alpha = (\mathbb{F}_2, \mathbb{F}_2, 0, 0)$.

Reidemeister moves in $\mathbb{RP}^2$