Twistings and Alexander polynomials

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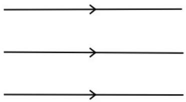
Motivations

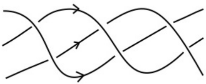
- There is a stabilization phenomenon for Khovanov homology of links obtained by inserting more and more full twists. Rozansky and Willis used this stabilization limit to define Khovanov homology for links in $\#^r S^1 \times S^2$.
- There is a similar stabilization phenomenon for $\widehat{\mathrm{HFK}}$ of links by inserting full twists on 2-strands by Peter Lambert-Cole.

A natural question to ask: Does similar result hold for $\widehat{\mathrm{HFK}}$ for inserting full twists on n-strands?

We give a positive answer at the level of Alexander polynomial.

Main result





Theorem (C.)

Let L be a link diagram, together with a choice of segment consisting of n parallel strands of the same orientation. Let L_m be the link obtained by inserting m full twists in the segment. The Alexander polynomial $\Delta_{L_m}(t)$ stabilizes as $m \to \infty$ in the following sense:
$\exists$ some Laurent series $h_L(t) \in \mathbb{C}[t^{-1}, t]]$ and an integer $r \in \left[\frac{n-1}{2}, n-1\right]$ such that the following holds:
$\forall k \in \mathbb{N}, \exists N \in \mathbb{N}$, such that $\forall m \geq N$, the first k terms of $\Delta_{L_m}(t)$ agree with the first k terms of

$$t^{mn(n-1-2r)/2} h_L(t).$$

Prototype examples: torus knots

Let $L = T(n, p)$ be the torus knot with the standard diagram.
Then $L_m = T(n, p + mn)$, with

$$\Delta_{L_m}(t) = t^{n-1/2} \frac{1-t}{1-t^n} \sum_{i=0}^{n-1} t^{(p+mn)(n-1-2i)/2}$$

When m is large, the first few terms are the same as the first few terms of

$$t^{(n-1)/2} \frac{1-t}{1-t^n} t^{-(n-1)(p+mn)/2},$$

so the limit Laurent series $h_L(t)$ is

$$h_L(t) = t^{-(p-1)(n-1)/2} \frac{1-t}{1-t^n} = t^{-(p-1)(n-1)/2}(1-t) \sum_{j=0}^{\infty} t^{jn},$$

and $r = n - 1$.

Slogan

"The Alexander polynomial of links behaves in the same manner with torus links under inserting full twists along parallel strands of the same orientation."

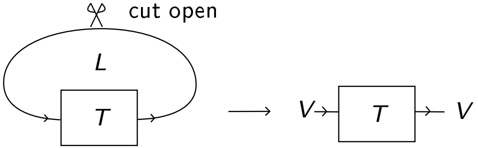
Now we briefly discuss the proof. The main tool is expressing $\Delta_L(t)$ as a $U_q(\mathfrak{gl}(1|1))$-invariant, described as in [Sartori'14].

$\Delta_L(t)$ as a $U_q(\mathfrak{gl}(1|1))$-invariant

The ribbon structure on the category of $U_q := U_q(\mathfrak{gl}(1|1))$-modules gives

$$\rho : \{\text{tangles}\} \to \{U_q\text{-equivariant maps}\}.$$

For a link L,



$$\rho(T) \in End_{U_q}(V) \cong \mathbb{C}(q) \cdot id_V, \quad \rho(T) = \Delta_L(t) \cdot id_V,$$

where $q = t^{1/2}$, and $V = L(\epsilon_1)$ is the vector space representation of U_q, a 2-dim highest weight module of weight ϵ_1.

Representation of the braid group $Br(n)$

Restricting ρ to n-strand braids oriented in the same direction, we get a representation of the braid group $Br(n)$:

$$\rho : Br(n) \to End_{U_q}(V^{\otimes n}).$$

Let τ denote the element representing the full twist in $Br(n)$.

Proposition

$$\rho(\tau^m) = \sum_{j=0}^{n-1} f_{m,j,n}(q)\rho(\tau^j)$$

for some Laurent polynomials $f_{m,j,n}(q)$.

We can give explicit formulae for $f_{m,j,n}(q)$, which stabilize as $m \to \infty$, and this leads to the stabilization result on the Alexander polynomial $\Delta_{L_m}(t)$.

Proof strategy for the proposition

- As a U_q-module, $V^{\otimes n}$ decomposes as a direct sum of irreducible U_q modules.
$$V^{\otimes n} = \bigoplus_{i=0}^{n-1} L((n-i)\epsilon_1 + i\epsilon_2)^{\oplus\binom{n-1}{i}}.$$
Denote $L((n-i)\epsilon_1 + i\epsilon_2)$ by L_i.
- Since the $Br(n)$ action is U_q-equivariant, we have
$$V^{\otimes n} = \bigoplus_{i=0}^{n-1} L_i \otimes H_i$$
as a U_q-$Br(n)$ bimodule, where U_q acts on L_i, and $Br(n)$ acts on H_i.
- By picking some basis, we construct an isomorphism $\phi_i : \wedge^i H_1 \to H_i$ as $Br(n)$-modules, up to some powers of q, e.g.
$$\phi_i(\sigma_j \cdot v) = q^{i-1}\sigma_j \cdot \phi_i(v).$$

Proof strategy, continued

- We can compute the action of full twist $\rho(\tau)|_{H_1}$ on H_1 , which is a scalar multiplication.
- By the above isomorphism, we get the full twist acts by scalar multiplication on each H_i, with different scalars for different H_i.
- Therefore, we can write the projections

$$\pi_i : V^{\otimes n} \to L_i \otimes H_i$$

as a $\mathbb{C}(q)$-linear combination of $\rho(\tau^0), ..., \rho(\tau^{n-1})$, by inverting a Vandermonde matrix. It follows we can express $\rho(\tau^m)$ in terms of $\rho(\tau^0), ..., \rho(\tau^{n-1})$ as well.

Schur-Weyl duality for U_q and $Br(n)$

The decomposition of $V^{\otimes n}$ as a U_q-$Br(n)$ bimodule

$$V^{\otimes n} \cong \bigoplus_{i=0}^{n-1} L_i \otimes H_i$$

is some statement of Schur-Weyl duality between U_q and $Br(n)$, where U_q takes the role of $GL(V)$, and $Br(n)$ takes the role of $Sym(n)$ in the usual Schur-Weyl duality.

Future work

- What happens if we change the orientation of some strands when inserting full twists? (obstruction: $V \otimes V^*$ is an indecomposable $U_q(\mathfrak{gl}(1|1))$-module)

- Does similar statement hold for $\widehat{\mathrm{HFK}}$?

Thank you!